On the smooth-fit property for one-dimensional optimal switching problem

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Abstract

This paper studies the problem of optimal switching for one-dimensional diffusion, which may be regarded as sequential optimal stopping problem with changes of regimes. The resulting dynamic programming principle leads to a system of variational inequalities, and the state space is divided into continuation regions and switching regions. By means of viscosity solutions approach, we prove the smoot-fit C^1 property of the value functions.

Key words : Optimal switching, system of variational inequalities, viscosity solutions, smooth-fit principle.

MSC Classification (2000) : 60G40, 49L25, 60H30.

1 Introduction

In this paper, we consider the optimal switching problem for a one dimensional stochastic process X. The diffusion process X may take a finite number of regimes that are switched at time decisions. The evolution of the controlled system is governed by

$$dX_t = b(X_t, I_t)dt + \sigma(X_t, I_t)dW_t,$$

with the indicator process of the regimes :

$$I_t = \sum_n \kappa_n 1_{\tau_n \leq t < \tau_{n+1}}.$$

Here W is a standard Bronian motion on a filtred probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, b, σ are given maps, $(\tau_n)_n$ is a sequence of increasing stopping times representing the switching regimes time decisions, and κ_n is $\mathcal{F}_{\tau_n}$-measurable valued in a finite set, representing the new chosen value of the regime at time τ_n and until τ_{n+1}.

Our problem consists in maximizing over the switching controls (τ_n, κ_n) the gain functional

$$E\left[\int_0^\infty e^{-\rho t} f(X_t, I_t)dt - \sum_n e^{-\rho \tau_n} g_{\kappa_{n-1},\kappa_n}\right]$$

where f is some running profit function depending on the current state and the regime, and g_{ij} is the cost for switching from regime i to j. We then denote by $v_i(x)$ the value function for this control problem when starting initially from state x and regime i.

Optimal switching problems for stochastic systems were studied by several authors, see [1], [4] or [7]. These control problems lead via the dynamic programming principle to a system of second order variational inequalities for the value functions v_i. Since the v_i are not smooth C^2 in general, a first mathematical point is to give a rigorous meaning to these variational PDE, either in Sobolev spaces as in [4], or by means of viscosity notion as in [7]. We also see that for each fixed regime i, the state space is divided into a switching region where it is optimal to change from regime i to some regime j, and the continuation region where it is optimal to stay in the current regime i. Optimal switching problem may be viewed as sequential optimal stopping problems with regimes shifts. It is well-known that optimal stopping problem leads to a free-boundary problem related to a variational inequality that divides the state space into the stopping region and the continuation region. Moreover, there is the so-called smooth-fit principle for optimal stopping problems that states the smoothness C^1 regularity of the value function through the boundary of the stopping region, once the reward function is smooth C^1 or is convex, see e.g. [6]. Smooth-fit principle for optimal stopping problems may be proved by different arguments and we mention recent ones in [2] or [5] based on local time and extended Itô's formula. Our main concern is to study such smooth-fit principle in the context of optimal switching problem, which has not yet been considered in the literature to the best of our knowledge.

Here, we use viscosity solutions arguments to prove the smooth-fit C^1 property of the value functions through the boundaries of the switching regions. The main difficulty with

regard to optimal stopping problems, comes from the fact that the switching region for the value function v_i depend also on the other value functions v_j for which one does not know a priori C^1 regularity (this is what we want to prove!) or convexity property. For this reason, it is an open question to see how extended Itô's formula and local time may be used to derive such smooth-fit property for optimal switching problems. Our proof arguments are relatively simple and does not require any specific knowledge on viscosity solutions theory.

The plan of this paper is organized as follows. In Section 2, we formulate our optimal switching problem and make some assumptions. Section 3 is devoted to the dynamic programming PDE characterization of the value functions by viscosity solutions, through a system of variational inequalities. In Section 4, we prove the smooth-fit property of the value functions.

2 Problem formulation and assumptions

We start with the mathematical framework for our optimal switching problem. The stochastic system X is valued in the state space $\mathcal{X} \subset \mathbb{R}$ assumed to be an interval with endpoints $-\infty \leq \ell < r \leq \infty$. We let $\mathbb{I}_d = \{1, \ldots, d\}$ the finite set of regimes. The dynamics of the controlled stochastic system is modeled as follows. We are given maps b, $\sigma : \mathcal{X} \times \mathbb{I}_d \rightarrow \mathbb{R}$ satisfying a Lipschitz condition in x :

(H1) $\quad |b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \leq C|x-y|, \quad \forall x, y \in \mathcal{X},\ i \in \mathbb{I}_d,$

for some positive constant C, and we require

(H2) $\quad \sigma(x,i) > 0, \quad \forall x \in \mathrm{int}(\mathcal{X}) \ = \ (\ell, r),\ i \in \mathbb{I}_d.$

We set $b_i(.) = b(., i)$, $\sigma_i(.) = \sigma(., i)$, $i \in \mathbb{I}_d$, and we assume that for any $x \in \mathcal{X}$, $i \in \mathbb{I}_d$, there exists a unique strong solution valued in $\mathcal{X}$ to the s.d.e.

$$dX_t \ = \ b_i(X_t)dt + \sigma_i(X_t)dW_t, \quad X_0 = x. \tag{2.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

A switching control α consists of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \kappa_1, \ldots, \kappa_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where τ_n are stopping times, $\tau_n < \tau_{n+1}$ and $\tau_n \rightarrow \infty$ a.s., and κ_n is $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$. We denote by $\mathcal{A}$ the set of all such switching controls. Now, for any initial condition $(x,i) \in \mathcal{X} \times \mathbb{I}_d$, and any control $\alpha = (\tau_n, \kappa_n)_{n\geq 1} \in \mathcal{A}$, there exists a unique strong solution valued in $\mathcal{X} \times \mathbb{I}_d$ to the controlled stochastic system :

$$X_0 \ = \ x, \quad I_{0^-} \ = \ i, \tag{2.2}$$

$$dX_t \ = \ b_{\kappa_n}(X_t)dt + \sigma_{\kappa_n}(X_t)dW_t, \quad I_t \ = \ \kappa_n, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0. \tag{2.3}$$

Here, we set $\tau_0 = 0$ and $\kappa_0 = i$. We denote by $(X^{x,i}, I^i)$ this solution (as usual, we omit the dependance in α for notational simplicity). We notice that $X^{x,i}$ is a continuous process and I^i is a cadlag process, possibly with a jump at time 0 if $\tau_1 = 0$ and so $I_0 = \kappa_1$.

We are given a running profit function $f : \mathcal{X} \times \mathbb{I}_d \rightarrow \mathbb{R}$, and we assume a Lipschitz condition :

(H3) $\qquad |f(x,i) - f(y,i)| \leq C|x-y|, \quad \forall x, y \in \mathcal{X},\ i \in \mathbb{I}_d,$

for some positive constant C. We also set $f_i(.) = f(.,i)$, $i \in \mathbb{I}_d$. The cost for switching from regime i to j is constant equal to g_{ij}. We assume that :

(H4) $\qquad 0 < g_{ik} \leq g_{ij} + g_{jk}, \quad \forall i \neq j \neq k \neq i \ \in \mathbb{I}_d.$

This last condition means that the switching cost is positive and it is no more expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j.

The expected total profit of running the system when initial state is (x,i) and using the switching control $\alpha = (\tau_n, \kappa_n)_{n\geq 1} \in \mathcal{A}$ is

$$J(x,i,\alpha) \;=\; E\left[\int_0^\infty e^{-\rho t} f(X_t^{x,i}, I_t^i)dt - \sum_{n=1}^{\infty} e^{-\rho\tau_n} g_{\kappa_{n-1},\kappa_n}\right],$$

where $\kappa_0 = i$. Here $\rho > 0$ is a positive discount factor, and we use the convention that $e^{-\rho\tau_n(\omega)} = 0$ when $\tau_n(\omega) = \infty$. We shall see below in Lemma 3.1 that the expectation defining $J(x,i,\alpha)$ is well-defined for ρ large enough (independent of x, i, α). The objective is to maximize this expected total profit over all strategies α. Accordingly, we define the function

$$v(x,i) \;=\; \sup_{\alpha\in\mathcal{A}} J(x,i,\alpha), \quad x \in \mathcal{X},\ i \in \mathbb{I}_d. \tag{2.4}$$

and we denote $v_i(.) := v(.,i)$ for $i \in \mathbb{I}_d$. The goal of this paper is to study the smoothness property of the value functions v_i. Our main result is the following :

Theorem 2.1 *Assume that* **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *hold. Then, for all $i \in \mathbb{I}_d$, the value function v_i is continuously differentiable on* $\mathrm{int}(\mathcal{X}) = (\ell, r)$.

3 Dynamic programming, viscosity solutions and system of variational inequalities

We first show the Lipschitz continuity of the value functions v_i.

Lemma 3.1 *Under* **(H1)** *and* **(H3)**, *there exists some positive constant $C > 0$ such that for all $\rho \geq C$, we have :*

$$|v_i(x) - v_i(y)| \;\leq\; C|x-y|, \quad \forall x, y \in \mathcal{X},\ i \in \mathbb{I}_d. \tag{3.1}$$

Proof. In the sequel, for notational simplicity, the C denotes a generic constant in different places, depending on the constants appearing in the Lipschitz conditions in **(H1)** and **(H3)**. For any $\alpha \in \mathcal{A}$, the solution to (2.2)-(2.3) is written as :

$$\begin{aligned} X_t^{x,i} &= x + \int_0^t b(X_s^{x,i}, I_s^i)ds + \int_0^t \sigma(X_s^{x,i}, I_s^i)dW_s \\ I_t^i &= \sum_{n=0}^{\infty} \kappa_n 1_{\tau_n \leq t < \tau_{n+1}}, \quad (\tau_0 = 0,\ \kappa_0 = i). \end{aligned}$$

By standard estimate for s.d.e. applying Itô's formula to $|X_t^{x,i}|^2$ and using Gronwall's lemma, we then obtain from the linear growth condition on b and σ in **(H1)** the following inequality for any $\alpha \in \mathcal{A}$:

$$
E\left|X_t^{x,i}\right|^2 \;\; \leq \;\; Ce^{Ct}(1+|x|^2), \quad t \geq 0.
$$

Hence, by linear growth condition on f in **(H3)**, this proves that for any $\alpha \in \mathcal{A}$:

$$
\begin{aligned}
E\left[\int_0^\infty e^{-\rho t}\left|f(X_t^{x,i}, I_t^i)\right| dt\right] \;&\leq\; CE\left[\int_0^\infty e^{-\rho t}(1+|X_t^{x,i}|)dt\right] \\
&\leq\; C\int_0^\infty e^{-\rho t}e^{Ct}(1+|x|)dt \\
&\leq\; C(1+|x|),
\end{aligned}
$$

for ρ larger than C. Recalling that the g_{ij} are nonnegative, this last inequality proves in particular that for all $(x,i,\alpha) \in \mathcal{X} \times \mathbb{I}_d \times \mathcal{A}$, $J(x,i,\alpha)$ is well-defined, valued in $[-\infty,\infty)$.

Moreover, by standard estimate for s.d.e. applying Itô's formula to $|X_t^{x,i} - X_t^{y,i}|^2$ and using Gronwall's lemma, we then obtain from the Lipschitz condition **(H1)** the following inequality uniformly in $\alpha \in \mathcal{A}$:

$$
E\left|X_t^{x,i} - X_t^{y,i}\right|^2 \;\; \leq \;\; e^{Ct}|x-y|^2, \quad \forall x,y \in \mathcal{X}, \; t \geq 0.
$$

From the Lipschitz condition **(H3)**, we deduce

$$
\begin{aligned}
|v_i(x) - v_i(y)| \;&\leq\; \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t}\left|f(X_t^{x,i}, I_t^i) - f(X_t^{y,i}, I_t^i)\right| dt\right] \\
&\leq\; C\sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t}\left|X_t^{x,i} - X_t^{y,i}\right| dt\right] \\
&\leq\; C\int_0^\infty e^{-\rho t}e^{Ct}|x-y|dt \;\leq\; C|x-y|,
\end{aligned}
$$

for ρ larger than C. This proves (3.1). □

In the rest of this paper, we shall now assume that ρ is large enough so that from the previous Lemma, the expected gain functional $J(x,i,\alpha)$ is well-defined for all x, i, α, and also the value functions v_i are continuous.

The dynamic programming principle is a well-known property in stochastic optimal control. In our optimal switching control problem, it is formulated as follows :

Dynamic programming principle : For any $(x,i) \in \mathcal{X} \times \mathbb{I}_d$, we have

$$
v(x,i) \;\; = \;\; \sup_{(\tau_n,\kappa_n)_n\in\mathcal{A}} E\left[\int_0^\theta e^{-\rho t}f(X_t^{x,i}, I_t^i)dt + e^{-\rho\theta}v(X_\theta^{x,i}, I_\theta^i) - \sum_{\tau_n\leq\theta} e^{-\rho\tau_n}g_{\kappa_{n-1},\kappa_n}\right], \tag{3.2}
$$

where θ is any stopping time, possibly depending on $\alpha \in \mathcal{A}$ in (3.2). This principle was formally stated in [1] and proved rigorously for the finite horizon case in [7]. The arguments for the infinite horizon case may be adapted in a straightforward way.

The dynamic programming principle combined with the notion of viscosity solutions are known to be a general and powerful tool for characterizing the value function of a stochastic control problem via a PDE representation, see [3]. We recall the definition of viscosity solutions for a P.D.E in the form

$$F(x, v, D_x v, D^2_{xx} v) \;=\; 0, \quad x \in \mathcal{O}, \tag{3.3}$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and F is a continuous function and noninceasing in its last argument (with respect to the order of symmetric matrices).

Definition 3.1 *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (3.3) *on $\mathcal{O}$ it it is*
(i) a viscosity supersolution to (3.3) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2 function φ in a neighborhood of x_0 s.t. x_0 is a local minimum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have :*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \;\geq\; 0.$$

and
(ii) a viscosity subsolution to (3.3) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2 function φ in a neighborhood of x_0 s.t. x_0 is a local maximum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have :*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \;\leq\; 0.$$

We shall denote by $\mathcal{L}_i$ the second order operator on the interior (ℓ, r) of $\mathcal{X}$ associated to the diffusion X solution to (2.1) :

$$\mathcal{L}_i\varphi \;=\; \frac{1}{2}\sigma_i^2\varphi\text{"} + b_i\varphi', \quad i \in \mathbb{I}_d.$$

Theorem 3.1 *Assume that* **(H1)** *and* **(H3)** *hold. Then, for each $i \in \mathbb{I}_d$, the value function v_i is a continuous viscosity solution on (ℓ, r) to the variational inequality :*

$$\min\left\{\rho v_i - \mathcal{L}_i v_i - f_i \,,\; v_i - \max_{j\neq i}(v_j - g_{ij})\right\} \;=\; 0, \quad x \in (\ell, r). \tag{3.4}$$

This means that for all $i \in \mathbb{I}_d$, we have both supersolution and subsolution properties :
(1) Viscosity supersolution property : for any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a local minimum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$, we have

$$\min\left\{\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x}) \,,\; v_i(\bar{x}) - \max_{j\neq i}(v_j - g_{ij})(\bar{x})\right\} \;\geq\; 0, \tag{3.5}$$

(2) Viscosity subsolution property : for any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a local maximum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$, we have

$$\min\left\{\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x}) \,,\; v_i(\bar{x}) - \max_{j\neq i}(v_j - g_{ij})(\bar{x})\right\} \;\leq\; 0, \tag{3.6}$$

Proof. The arguments of this proof are standard, based on the dynamic programming principle and Itô's formula. We defer the proof in the appendix. □

For any regime $i \in \mathbb{I}_d$, we introduce the switching region :

$$\mathcal{S}_i \;=\; \left\{ x \in (\ell, r) \;: v_i(x) = \max_{j \neq i} (v_j - g_{ij})(x) \right\}.$$

$\mathcal{S}_i$ is a closed subset of (ℓ, r) and corresponds to the region where it is optimal to change of regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in (ℓ, r) is the so-called continuation region :

$$\mathcal{C}_i \;=\; \left\{ x \in (\ell, r) \;: v_i(x) > \max_{j \neq i} (v_j - g_{ij})(x) \right\},$$

where one remains in regime i.

Remark 3.1 Let us consider the following optimal stopping problem :

$$v(x) \;=\; \sup_{\tau \text{ stopping times}} E\left[\int_0^\tau e^{-\rho\tau} f(X_t^x) dt + e^{-\rho\tau} h(X_\tau^x) \right]. \tag{3.7}$$

It is well-know that the dynamic programming principle for (3.7) leads to a variational inequality for v in the form :

$$\min \{ \rho v - \mathcal{L} v - f \;,\; v - h \} \;=\; 0,$$

where $\mathcal{L}$ is the infinitesimal generator of the diffusion X. Moreover, the state space domain of X is divided into the stopping region

$$\mathcal{S} \;=\; \{ x \;: v(x) = h(x) \},$$

and its complement set, the continuation region :

$$\mathcal{C} \;=\; \{ x \;: v(x) > h(x) \}.$$

The smooth-fit principle for optimal stopping problems states that the value function v is smooth C^1 through the boundary of the stopping region, the so-called free boundary, once h is C^1 or convex.

Our aim is to state similar results for optimal switching problems. The main difficulty comes from the fact that we have a system a variational inequalities, so that the switching region for v_i depend also on the other value functions v_j which are not convex or known to be C^1 a priori.

4 The smooth-fit property

We first show, like for optimal stopping problems, that the value functions are smooth C^2 in their continuation regions. We provide here a quick proof based on viscosity solutions arguments.

Lemma 4.1 *Assume that* **(H1)**, **(H2)** *and* **(H3)** *hold. Then, for all $i \in \mathbb{I}_d$, the value function v_i is smooth C^2 on $\mathcal{C}_i$ and satisfies in a classical sense :*

$$\rho v_i(x) - \mathcal{L}_i v_i(x) - f_i(x) \;=\; 0, \quad x \in \mathcal{C}_i. \tag{4.1}$$

Proof. We first check that v_i is a viscosity solution to (4.1). Let $\bar{x} \in \mathcal{C}_i$ and φ a C^2 function on $\mathcal{C}_i$ s.t. $\bar{x}$ is a local maximum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$. Then, by definition of $\mathcal{C}_i$, we have $v_i(\bar{x}) > \max_{j\neq i}(v_j - g_{ij})(\bar{x})$, and so from the subsolution viscosity property (3.6) of v_i, we have :

$$\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x}) \;\leq\; 0.$$

The supersolution inequality for (4.1) is immediate from (3.5).

Now, for arbitrary bounded interval $(x_1, x_2) \subset \mathcal{C}_i$, consider the Dirichlet boundary linear problem :

$$\rho w(x) - \mathcal{L}_i w(x) - f_i(x) \;=\; 0, \quad \text{on } (x_1, x_2) \tag{4.2}$$

$$w(x_1) \;=\; v_i(x_1), \qquad w(x_2) \;=\; v_i(x_2). \tag{4.3}$$

Under the nondegeneracy condition **(H2)**, classical results provide the existence and uniqueness of a smooth C^2 function w solution on (x_1, x_2) to (4.2)-(4.3). In particular, this smooth function w is a viscosity solution of (4.1) on (x_1, x_2). From standard uniqueness results on viscosity solutions (here for a linear PDE in a bounded domain), we deduce that $v_i = w$ on (x_1, x_2). From the arbitrariness of $(x_1, x_2) \subset \mathcal{C}_i$, this proves that v_i is smooth C^2 on $\mathcal{C}_i$, and so satisfies (4.1) in a classical sense. □

We now state an elementary partition property on the switching regions.

Lemma 4.2 *Assume that* **(H1)**, **(H3)** *and* **(H4)** *hold. Then, for all $i \in \mathbb{I}_d$, we have $\mathcal{S}_i = \cup_{j\neq i}\mathcal{S}_{ij}$ where*

$$\mathcal{S}_{ij} \;=\; \{x \in \mathcal{C}_j \; : v_i(x) = (v_j - g_{ij})(x)\}.$$

Proof. Denote $\tilde{\mathcal{S}}_i = \cup_{j\neq i}\mathcal{S}_{ij}$. Since we always have $v_i \geq \max_{j\neq i}(v_j - g_{ij})$, the inclusion $\tilde{\mathcal{S}}_i \subset \mathcal{S}_i$ is clear.

Conversely, let $x \in \mathcal{S}_i$. Then there exists $j \neq i$ s.t. $v_i(x) = v_j(x) - g_{ij}$. We have two cases :

⋆ if x lies in $\mathcal{C}_j$, then $x \in \mathcal{S}_{ij}$ and so $x \in \tilde{\mathcal{S}}_i$.

⋆ if x does not lie in $\mathcal{C}_j$, then x would lie in $\mathcal{S}_j$, which means that one could find some $k \neq j$ s.t. $v_j(x) = v_k(x) - g_{jk}$, and so $v_i(x) = v_k(x) - g_{ij} - g_{jk}$. From condition **(H4)** and since we always have $v_i \geq v_k - g_{ik}$, this would imply $v_i(x) = v_k(x) - g_{ik}$. Since cost g_{ik} is positive, we also get that $k \neq i$. Again, we have two cases : if x lies in $\mathcal{C}_k$, then x lies in $\mathcal{S}_{ik}$ and so in $\tilde{\mathcal{S}}_i$. Otherwise, we repeat the above argument and since the number of states is finite, we should necessarily find some $l \neq i$ s.t. $v_i(x) = v_l(x) - g_{il}$ and $x \in \mathcal{C}_l$. This shows finally that $x \in \tilde{\mathcal{S}}_i$. □

Remark 4.1 $\mathcal{S}_{ij}$ represents the region where it is optimal to switch from regime i to regime j and stay here for a moment, i.e. without changing instantaneously from regime j to another regime.

We can finally prove the smooth-fit property of the value functions v_i through the boundaries of the switching regions.

Theorem 4.1 *Assume that* **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *hold. Then, for all* $i \in \mathbb{I}_d$, *the value function* v_i *is continuously differentiable on* (ℓ, r). *Moreover, at* $x \in \mathcal{S}_{ij}$, *we have* $v_i'(x) = v_j'(x)$.

Proof. We already know from Lemma 4.1 that v_i is smooth C^2 on the open set $\mathcal{C}_i$ for all $\in \mathbb{I}_d$. We have to prove the C^1 property of v_i at any point of the closed set $\mathcal{S}_i$. We denote for all $j \in \mathbb{I}_d$, $j \neq i$, $h_j = v_j - g_{ij}$ and we notice that h_j is smooth C^1 (actually even C^2) on $\mathcal{C}_j$.

1. We first check that v_i admits a left and right derivative $v'_{i,-}(x_0)$ and $v'_{i,+}(x_0)$ at any point x_0 in $\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}$. We distinguish the two following cases :

• *Case a)* x_0 lies in the interior $\mathrm{Int}(\mathcal{S}_i)$ of $\mathcal{S}_i$. Then, we have two subcases :

$\star$ $x_0 \in \mathrm{Int}(\mathcal{S}_{ij})$ for some $j \neq i$, i.e. there exists some $\delta > 0$ s.t. $[x_0 - \delta, x_0 + \delta] \subset \mathcal{S}_{ij}$. By definition of $\mathcal{S}_{ij}$, we then have $v_i = h_j$ on $[x_0 - \delta, x_0 + \delta] \subset \mathcal{C}_j$, and so v_i is differentiable at x_0 with $v_i'(x_0) = h_j'(x_0)$.

$\star$ There exists $j \neq k \neq i$ in $\mathbb{I}_d$ and $\delta > 0$ s.t. $[x_0 - \delta, x_0] \subset \mathcal{S}_{ij}$ and $[x_0, x_0 + \delta] \subset \mathcal{S}_{ik}$. We then have $v_i = h_j$ on $[x_0 - \delta, x_0] \subset \mathcal{C}_j$ and $v_i = h_k$ on $[x_0, x_0 + \delta] \subset \mathcal{C}_k$. Thus, v_i admits a left and right derivative at x_0 with $v'_{i,-}(x_0) = h_j'(x_0)$ and $v'_{i,+}(x_0) = h_k'(x_0)$.

• *Case b)* x_0 lies in the boundary $\partial\mathcal{S}_i = \mathcal{S}_i \setminus \mathrm{Int}(\mathcal{S}_i)$ of $\mathcal{S}_i$. We assume that x_0 lies in the left-boundary of $\mathcal{S}_i$, i.e. there exists $\delta > 0$ s.t. $[x_0 - \delta, x_0) \subset \mathcal{C}_i$ (the other case where x_0 lies in the right-boundary is dealt with similarly). Recalling that on $\mathcal{C}_i$, v_i is solution to : $\rho v_i - \mathcal{L} v_i - f_i = 0$, we deduce that on $[x_0 - \delta, x_0)$, v_i is equal to w_i the unique smooth C^2 solution to the o.d.e. : $\rho w_i - \mathcal{L} w_i - f_i = 0$ with the boundaries conditions : $w_i(x_0 - \delta) = v_i(x_0 - \delta)$, $w_i(x_0) = v_i(x_0)$. Therefore, v_i admits a left derivative at x_0 with $v'_{i,-}(x_0) = w_i'(x_0)$. In order to prove that v_i admits a right derivative, we distinguish the two subcases :

$\star$ There exists $j \neq i$ in $\mathbb{I}_d$ and $\delta' > 0$ s.t. $[x_0, x_0 + \delta'] \subset \mathcal{S}_{ij}$. Then, on $[x_0, x_0 + \delta']$, v_i is equal to h_j. Hence v_i admits a right derivative at x_0 with $v'_{i,+}(x_0) = h_j'(x_0)$.

$\star$ Otherwise, for all $j \neq i$, we can find a sequence (x_n^j) s.t. $x_n^j \geq x_0$, $x_n^j \notin \mathcal{S}_{ij}$ and $x_n^j \to x_0$. By a diagonalization procedure, we construct then a sequence (x_n) s.t. $x^n \geq x_0$, $x_n \notin \mathcal{S}_{ij}$ for all $j \neq i$, i.e. $x_n \in \mathcal{C}_i$, and $x_n \to x_0$. Since $\mathcal{C}_i$ is open, there exists then $\delta'' > 0$ s.t. $[x_0, x_0 + \delta''] \subset \mathcal{C}_i$. We deduce that on $[x_0, x_0 + \delta'']$, v_i is equal to $\hat{w}_i$ the unique smooth C^2 solution to the o.d.e. $\rho \hat{w}_i - \mathcal{L} \hat{w}_i - f_i = 0$ with the boundaries conditions $\hat{w}_i(x_0) = v_i(x_0)$, $\hat{w}_i(x_0 + \delta'') = v_i(x_0 + \delta'')$. In particular, v_i admits a right derivative at x_0 with $v'_{i,+}(x_0) = \hat{w}_i'(x_0)$.

2. Consider now some point in $\mathcal{S}_i$ eventually on its boundary. We recall again that from Lemma 4.2, there exists some $j \neq i$ s.t. $x_0 \in \mathcal{S}_{ij}$: $v_i(x_0) = h_j(x_0)$, and h_j is smooth C^1on

x_0 in $\mathcal{C}_j$. Since $v_j \geq h_j$, we deduce that

$$\begin{aligned}
\frac{v_i(x) - v_i(x_0)}{x - x_0} &\leq \frac{h_j(x) - h_j(x_0)}{x - x_0}, \quad \forall\, x < x_0 \\
\frac{v_i(x) - v_i(x_0)}{x - x_0} &\geq \frac{h_j(x) - h_j(x_0)}{x - x_0}, \quad \forall\, x > x_0,
\end{aligned}$$

and so :

$$v'_{i,-}(x_0) \leq h'_j(x_0) \leq v'_{i,+}(x_0).$$

We argue by contradiction and suppose that v_i is not differentiable at x_0. Then, in view of the above inequality, one can find some $p \in (v'_{i,-}(x_0), v'_{i,+}(x_0))$. Consider, for $\varepsilon > 0$, the smooth C^2 function :

$$\varphi_\varepsilon(x) = v_i(x_0) + p(x - x_0) + \frac{1}{2\varepsilon}(x - x_0)^2.$$

Then, we see that v_i dominates locally in a neighborhood of x_0 the function φ_ε, i.e x_0 is a local minimum of $v_i - \varphi_\varepsilon$. From the supersolution viscosity property of v_i to the PDE (3.4), this yields :

$$\rho\varphi_\varepsilon(x_0) - \mathcal{L}_i\varphi_\varepsilon(x_0) - f_i(x_0) \geq 0,$$

which is written as :

$$\rho v_i(x_0) - b_i(x_0)p - f_i(x_0) - \frac{1}{2\varepsilon}\sigma_i^2(x_0) \geq 0.$$

Sending ε to zero provides the required contradiction under **(H2)**. We have then proved that for $x_0 \in \mathcal{S}_{ij}$, $v'_i(x_0) = h'_j(x_0) = v'_j(x_0)$. □

Appendix: Proof of Theorem 3.1

(1) *Viscosity supersolution property.*

Fix $i \in \mathbb{I}_d$. Consider any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a minimum of $v_i - \varphi$ in a neighborhood $B_\varepsilon(\bar{x}) = (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ of $\bar{x}$, $\varepsilon > 0$, and $v_i(\bar{x}) = \varphi(\bar{x})$. By taking the immediate switching control $\tau_1 = 0$, $\kappa_1 = j \neq i$, $\tau_n = \infty$, $n \geq 2$, and $\theta = 0$ in the relation (3.2), we obtain

$$v_i(\bar{x}) \geq v_j(\bar{x}) - g_{ij}, \quad \forall j \neq i. \tag{A.1}$$

On the other hand, by taking the no-switching control $\tau_n = \infty$, $n \geq 1$, i.e. $I_t^i = i$, $t \geq 0$, $X^{\bar{x},i}$ stays in regime i with diffusion coefficients b_i and σ_i, and $\theta = \tau_\varepsilon \wedge h$, with $h > 0$ and $\tau_\varepsilon = \inf\{t \geq 0 \ : X_t^{\bar{x},i} \notin B_\varepsilon(\bar{x})\}$, we get from (3.2) :

$$\begin{aligned}
\varphi(\bar{x}) = v_i(\bar{x}) &\geq E\left[\int_0^\theta e^{-\rho t} f_i(X_t^{\bar{x},i})dt + e^{-\rho\theta} v_i(X_\theta^{\bar{x},i})\right] \\
&\geq E\left[\int_0^\theta e^{-\rho t} f_i(X_t^{\bar{x},i})dt + e^{-\rho\theta} \varphi(X_\theta^{\bar{x},i})\right]
\end{aligned}$$

By applying Itô's formula to $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $\theta = \tau_\varepsilon \wedge h$ and plugging into the last inequality, we obtain :

$$\frac{1}{h} E\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t}\left(\rho\varphi - \mathcal{L}_i\varphi - f_i\right)(X_t^{\bar{x},i})\right] \;\; \geq \;\; 0.$$

From the dominated convergence theorem, this yields by sending h to zero :

$$(\rho\varphi - \mathcal{L}_i\varphi - f_i)(\bar{x}) \;\; \geq \;\; 0.$$

By combining with (A.1), we obtain the required supersolution inequality (3.5).

(2) *Viscosity subsolution property.*

Fix $i \in \mathbb{I}_d$, and consider any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a maximum of $v_i - \varphi$ in a neighborhood $B_\varepsilon(\bar{x}) = (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ of $\bar{x}$, $\varepsilon > 0$, and $v_i(\bar{x}) = \varphi(\bar{x})$. We argue by contradiction by assuming on the contrary that (3.6) does not hold so that by continuity of v_i, v_j, $j \neq i$, φ and its derivatives, there exists some $0 < \delta \leq \varepsilon$ s.t.

$$(\rho\varphi - \mathcal{L}_i\varphi - f_i)(x) \;\; \geq \;\; \delta, \quad \forall x \in B_\delta(\bar{x}) = (x - \delta, x + \delta) \tag{A.2}$$

$$v_i(x) \; - \; \max_{j \neq i}(v_j - g_{ij})(x) \;\; \geq \;\; \delta, \quad \forall x \in B_\delta(\bar{x}). \tag{A.3}$$

For any $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$, consider the exit time $\tau_\delta = \inf\{t \geq 0 \;:\; X_t^{\bar{x},i} \notin B_\delta(\bar{x})\}$. By applying Itô's formula to $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $\theta = \tau_1 \wedge \tau_\delta$, we have by noting that before θ, $X^{x,i}$ stays in regime i and in the ball $B_\delta(\bar{x}) \subset B_\varepsilon(\bar{x})$:

$$\begin{aligned} v_i(\bar{x}) \; = \; \varphi(\bar{x}) \;\; &= \;\; E\left[\int_0^\theta e^{-\rho t}(\rho\varphi - \mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt + e^{-\rho\theta}\varphi(X_\theta^{\bar{x},i})\right] \\ &\geq \;\; E\left[\int_0^\theta e^{-\rho t}(\rho\varphi - \mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt + e^{-\rho\theta}v_i(X_\theta^{\bar{x},i})\right]. \end{aligned} \tag{A.4}$$

Now, since $\theta = \tau_\delta \wedge \tau_1$, we have

$$\begin{aligned} e^{-\rho\theta}v(X_\theta^{\bar{x},i}, I_\theta^i) - \sum_{\tau_n \leq \theta} g_{\kappa_{n-1},\kappa_n} \;\; &= \;\; e^{-\rho\tau_1}\left(v(X_{\tau_1}^{\bar{x},i}, \kappa_1) - g_{i\kappa_1}\right)1_{\tau_1 \leq \tau_\delta} + e^{-\rho\tau_\delta}v_i(X_{\tau_\delta}^{\bar{x},i})1_{\tau_\delta < \tau_1} \\ &\leq \;\; e^{-\rho\tau_1}\left(v_i(X_{\tau_1}^{\bar{x},i}) - \delta\right)1_{\tau_1 \leq \tau_\delta} + e^{-\rho\tau_\delta}v_i(X_{\tau_\delta}^{\bar{x},i})1_{\tau_\delta < \tau_1} \\ &= \;\; e^{-\rho\theta}v_i(X_\theta^{\bar{x},i}) - \delta e^{-\rho\tau_1}1_{\tau_1 \leq \tau_\delta}, \end{aligned}$$

where the inequality follows from (A.3). By plugging into (A.4) and using (A.2), we get :

$$\begin{aligned} v_i(\bar{x}) \;\; \geq \;\; & E\left[\int_0^\theta e^{-\rho t}f_i(X_t^{\bar{x},i})dt + e^{-\rho\theta}v(X_\theta^{\bar{x},i}, I_\theta^i) - \sum_{\tau_n \leq \theta} g_{\kappa_{n-1},\kappa_n}\right] \\ & + \delta \, E\left[\int_0^\theta e^{-\rho t}dt + e^{-\rho\tau_1}1_{\tau_1 \leq \tau_\delta}\right]. \end{aligned} \tag{A.5}$$

We now claim that there exists some positive constant $c_0 > 0$ s.t. :

$$E\left[\int_0^\theta e^{-\rho t}dt + e^{-\rho\tau_1}1_{\tau_1 \leq \tau_\delta}\right] \;\; \geq \;\; c_0, \quad \forall \alpha \in \mathcal{A}.$$

For this, we construct a smooth function w s.t.

$$
\begin{aligned}
\max\{\rho w(x) - \mathcal{L}_i w(x) - 1\ ,\ w(x) - 1\} &\leq 0,\ \forall x \in B_\delta(\bar{x}) &\text{(A.6)}\\
w(x) &= 0,\ \forall x \in \partial B_\delta(\bar{x}) = \{x\ : |x - \bar{x}| = \delta\} &\text{(A.7)}\\
w(\bar{x}) &> 0. &\text{(A.8)}
\end{aligned}
$$

For instance, we can take the function $w(x) = c_0\left(1 - \frac{|x-\bar{x}|^2}{\delta^2}\right)$, with

$$
0 < c_0 \quad \leq \quad \min\left\{\left(\rho + \frac{2}{\delta}\sup_{x\in B_\delta(\bar{x})}|b_i(x)| + \frac{1}{\delta^2}\sup_{x\in B_\delta(\bar{x})}|\sigma_i(x)|^2\right)^{-1}\ ,\ 1\right\}.
$$

Then, by applying Itô's formula to $e^{-\rho t}w(X_t^{\bar{x},i})$ between 0 and $\theta = \tau_\delta \wedge \tau_1$, we have :

$$
\begin{aligned}
0\ <\ c_0\ =\ w(\bar{x}) &= E\left[\int_0^\theta e^{-\rho t}(\rho w - \mathcal{L}_i w)(X_t^{\bar{x},i})dt + e^{-\rho\theta}w(X_\theta^{\bar{x},i})\right]\\
&\leq E\left[\int_0^\theta e^{-\rho t}dt + e^{-\rho\tau_1}1_{\tau_1\leq\tau_\delta}\right],
\end{aligned}
$$

from (A.6), (A.7) and (A.8). By plugging this last inequality (uniform in α) into (A.5), we then obtain :

$$
v_i(\bar{x}) \quad \geq \quad \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\theta e^{-\rho t}f_i(X_t^{\bar{x},i})dt + e^{-\rho\theta}v(X_\theta^{\bar{x},i}, I_\theta^i) - \sum_{\tau_n\leq\theta} g_{\kappa_{n-1},\kappa_n}\right] \ +\ \delta c_0,
$$

which is in contradiction with the dynamic programming principle (3.2).

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